8-13:

Just finished Integration in multiple variables

- Incoming: Recalling vectors and parametrized curves/surfaces
  - Will discover deep relationships between Integration and vector fields
- Idea: How do we find (and work with) interesting geometric quantities?

- Recall: vectors

For us, vectors $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$

$$= \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \langle a, b, c \rangle$$

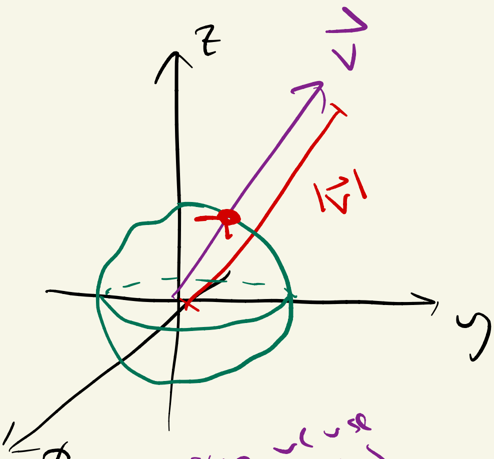
Coordinate representations

- Great for computations

Alternative (geometric approach)

Vectors have two "features":

- Magnitude (size!)
  - $|\vec{v}| = \sqrt{a^2+b^2+c^2}$
  - Scalar



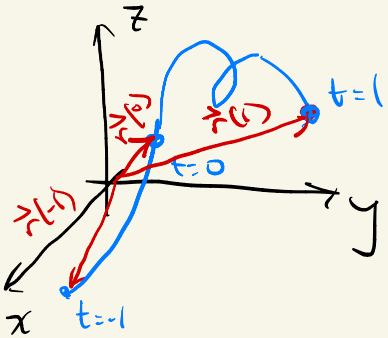
- Direction
  - Unit vector = point on unit sphere

often, we use "hats" (like $\hat{i}$) for unit vectors

vector (using scalar mult. with $\frac{1}{|\vec{v}|}$)

$$\frac{\vec{v}}{|\vec{v}|} = \frac{a}{|\vec{v}|}\hat{i} + \frac{b}{|\vec{v}|}\hat{j} + \frac{c}{|\vec{v}|}\hat{k}$$

Parametrized Curves



- 1D object (1 degree of freedom, forwards & backwards) ✓ since it depends only on 1 real number, t

- For each $t \in \mathbb{R}$ (or line segment), we get a vector $\vec{r}(t)$

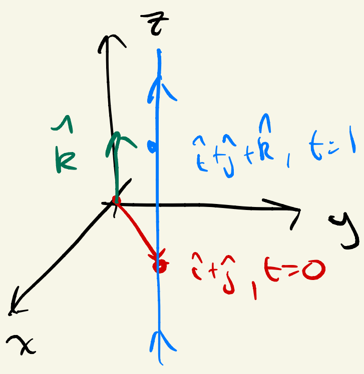
  Ex: position of a particle

- $\vec{r}(t) = a(t)\hat{i} + b(t)\hat{j} + c(t)\hat{k}$

  functions of t (if position, t = time)

- $= \begin{bmatrix} a(t) \\ b(t) \\ c(t) \end{bmatrix}$

Ex: Line



$$\vec{r}(t) = \hat{i} + \hat{j} + t\hat{k}$$

$$= \underbrace{(\hat{i}+\hat{j})}_{\text{part independent of } t \text{ (constant term)}} + \underbrace{t(\hat{k})}_{\text{depends linearly on } t}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Look: $\vec{r}(0) = \hat{i} + \hat{j}$

$\vec{r}(1) = \hat{i} + \hat{j} + 1\hat{k}$

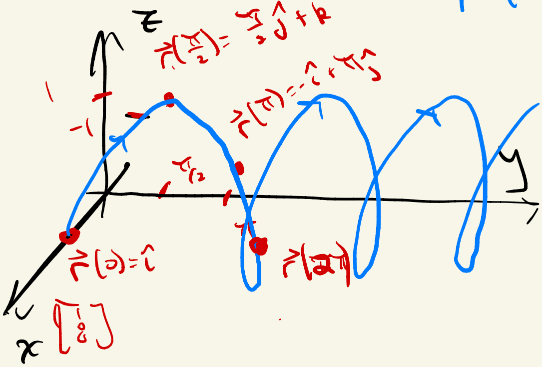
$= \hat{i} + \hat{j} + \hat{k}$

Ex: Helix

$$\vec{r}(t) = \underbrace{\cos t\,\hat{i}}_{} + t\,\hat{j} + \underbrace{\sin t\,\hat{k}}_{}$$

suggests linear motion

suggest "circular" motion



$$\vec{r}(0) = \hat{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\vec{r}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2}\hat{j} + \hat{k} = \begin{bmatrix}0\\ \pi/2\\ 1\end{bmatrix}$$

$$\vec{r}(\pi) = -\hat{i} + \pi\hat{j} = \begin{bmatrix}-1\\ \pi\\ 0\end{bmatrix}$$

$$\vec{r}(2\pi) = \hat{i} + 2\pi\hat{j} = \begin{bmatrix}1\\ 2\pi\\ 0\end{bmatrix}$$

Now, if $\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ is position of a particle:

- Derivative ("velocity")

$$\vec{v} = \frac{d\vec{r}}{dt}(t) = \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}$$

- Note: speed $= |\vec{v}|$, magnitude.
  can have constant speed, but variable velocity:
  e.g. circular motion $\vec{v}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$
  constant speed: $|\vec{v}| = \sqrt{\cos^2 t + \sin^2 t} = 1$

- 2^{nd} Derivative ("acceleration")

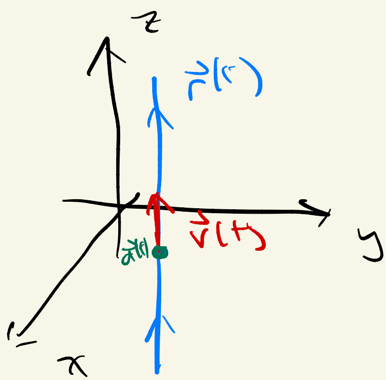
$$\vec{a} = \frac{d^2\vec{r}}{dt^2}(t) = \frac{d^2 f}{dt^2}\hat{i} + \frac{d^2 g}{dt^2}\hat{j} + \frac{d^2 h}{dt^2}\hat{k}$$

Ex: Line

$\vec{r}(t) = \hat{i} + \hat{j} + t\hat{k}$

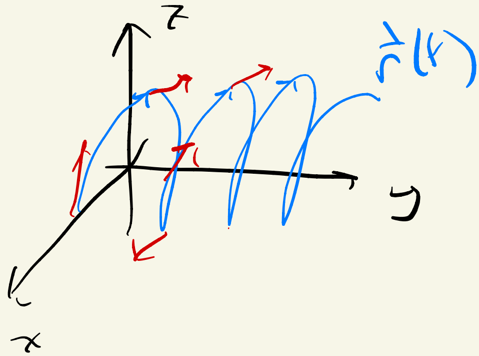
$\vec{v}(t) = \hat{k}$

$\vec{a}(t) = \vec{0}$



"Moves at constant speed"

Ex: Helix

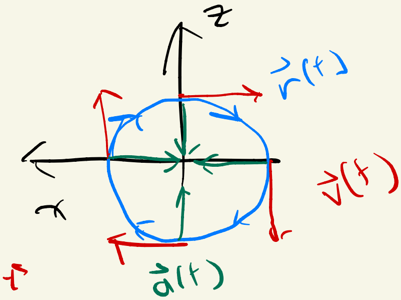


$\vec{r}(t) = \cos t\,\hat{i} + t\,\hat{j} + \sin t\,\hat{k}$

$\vec{v}(t) = -\sin t\,\hat{i} + \hat{j} + \cos t\,\hat{k}$

$\vec{a}(t) = -\cos t\,\hat{i} - \sin t\,\hat{k}$

These affect magnitude and direction. Special things happen when either is constant



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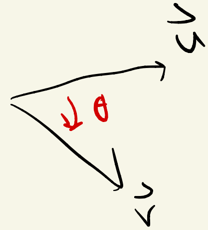
This always happens →

- $\vec{v}(t)$ is tangent to $\vec{r}(t)$
- $\vec{a}(t)$ is perpendicular to $\vec{v}(t)$
  - This is because $\vec{v}(t)$ has constant speed

"Multiplication"

we can't directly multiply vectors (that only makes sense for scalars).

Analogues of multiplication:



Two vectors ↓ scalar!

• Dot product: $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$

$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

$$\rightarrow \quad \vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cos\theta \quad \leftarrow \text{very geometric}$$

$$= u_1v_1 + u_2v_2 + u_3v_3$$

Useful fact:

$$\vec{u}\cdot\vec{u} = u_1^2 + u_2^2 + u_3^2$$

$$= |\vec{u}|^2$$

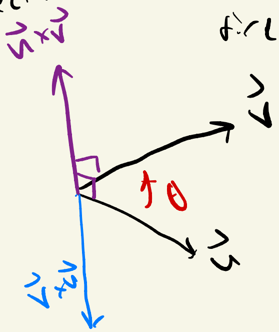
$$\Rightarrow \quad |\vec{u}| = \sqrt{\vec{u}\cdot\vec{u}}$$

• Cross product:

Two vectors ↓ vector

Magnitude: $|\vec{u}\times\vec{v}| = |\vec{u}|\cdot|\vec{v}|\cdot\sin\theta$

Direction: $\vec{u}\times\vec{v}$ is perpendicular to $\vec{u}$ and $\vec{v}$, direction chosen by right hand rule



Notice: $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

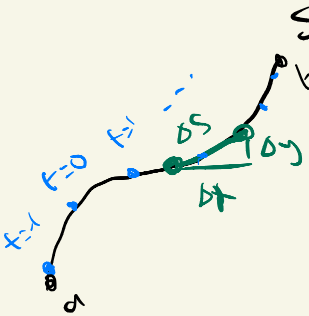
Not commutative

Product rule (Leibniz rule): (implicitly, $\vec{u}(t)$ and $\vec{v}(t)$ are parametrized curves)

$$\frac{d}{dt}\left(\vec{u}\cdot\vec{v}\right) = \frac{d\vec{u}}{dt}\cdot\vec{v} + \vec{u}\cdot\frac{d\vec{v}}{dt}$$

$$\frac{d}{dt}\left(\vec{u}\times\vec{v}\right) = \frac{d\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{d\vec{v}}{dt}$$

How do we find length of parametrized curves?



if parametrized, can talk about Δt

$$\Delta s \approx \sqrt{\Delta x^2 + \Delta y^2}$$

$$= \left(\sqrt{\frac{\Delta x^2}{\Delta t^2} + \frac{\Delta y^2}{\Delta t^2}}\right)\cdot \Delta t$$

$$\Rightarrow \quad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\cdot dt$$

Then we take small time limit, $\Delta t \to 0$, and add them all up...

length $\rightarrow$ $$L = \int_a^b ds$$

after parametrizing $\rightarrow$ $$= \int_{t_a}^{t_b} \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}_{=|\vec{v}|}\cdot dt$$

$$\Rightarrow \boxed{L = \int_{t_a}^{t_b} |\vec{v}|\, dt}$$

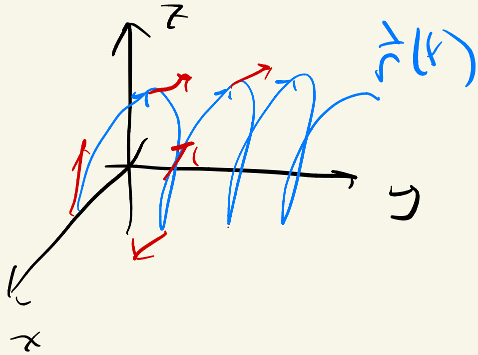
arc length

Ex: Helix

$$\vec{r}(t) = \cos t\,\hat{i} + t\,\hat{j} + \sin t\,\hat{k}$$

$$\vec{v}(t) = -\sin t\,\hat{i} + \hat{j} + \cos t\,\hat{k}$$

$$\vec{a}(t) = -\cos t\,\hat{i} - \sin t\,\hat{k}$$



Distance travelled from $t=0$ to 2π?

$$L = \int_0^{2\pi} |\vec{v}|\,dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + 1 + (\cos t)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{2}\,dt$$

$$= 2\sqrt{2}\,\pi$$

But wait... $ds = \sqrt{dx^2 + dy^2}$ doesn't depend upon a parameterization (no dt dependence: cancels by chain rule)

and in some sense represents the more fundamental idea of "pieces of length"

We know:

$$\frac{ds}{dt} = |\vec{v}|$$

$\frac{ds}{dt}$ is speed, not velocity, because it just catches magnitude, not direction.

What if we wanted to know how position vector $\vec{r}$ changes with respect to arc length s? (arc length s can be thought of as a function of t!)

Well chain rule says:

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \vec{v} \cdot \frac{1}{|\vec{v}|} = \frac{\vec{v}}{|\vec{v}|}$$

only works if $\frac{ds}{dt} \neq 0$

unit vector direction of $|\vec{v}|$

This is special! Let's give it a name

Def: Unit Tangent vector = direction of velocity vector

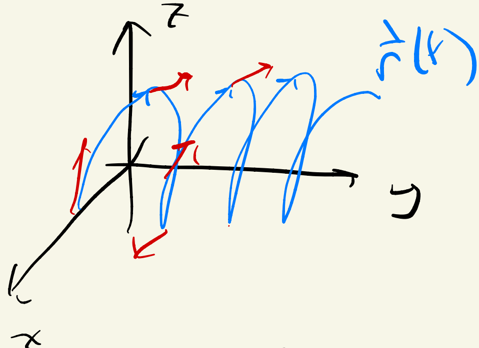
$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} \qquad (\text{implicitly, } \vec{v} = \vec{v}(t))$$

Ex: Helix

$$\vec{r}(t) = \cos t\,\hat{i} + t\,\hat{j} + \sin t\,\hat{k}$$

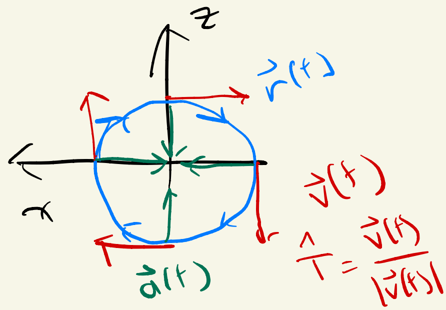
$$\vec{v}(t) = -\sin t\,\hat{i} + \hat{j} + \cos t\,\hat{k}$$

$$\vec{a}(t) = -\cos t\,\hat{i} - \sin t\,\hat{k}$$



$$\hat{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{-\sin t\,\hat{i} + \hat{j} + \cos t\,\hat{k}}{\sqrt{(-\sin t)^2 + 1^2 + (\cos t)^2}}$$

$$= \frac{-\sin t}{\sqrt{2}}\,\hat{i} + \frac{1}{\sqrt{2}}\,\hat{j} + \frac{\cos t}{\sqrt{2}}\,\hat{k}$$



- Well, if we have unit tangent vector which measures direction of $\vec{v}$ ... why not a unit vector measuring $\vec{T}$'s change in direction?

Def: Unit Normal

$$\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}$$

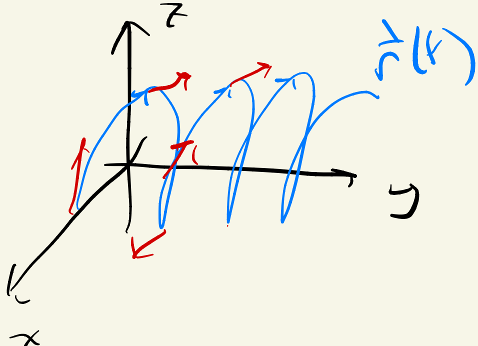
unit vector in direction of rate of change of $\vec{T}$

Ex: Helix

$\vec{r}(t) = \cos t\,\hat{i} + t\,\hat{j} + \sin t\,\hat{k}$

$\vec{v}(t) = -\sin t\,\hat{i} + \hat{j} + \cos t\,\hat{k}$

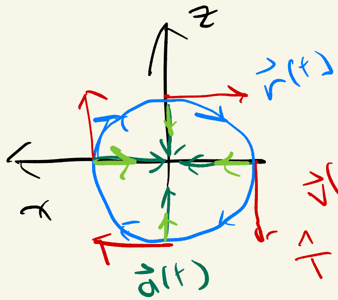
$\vec{a}(t) = -\cos t\,\hat{i} - \sin t\,\hat{k}$



$$\hat{T} = \frac{-\sin t}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{\cos t}{\sqrt{2}}\hat{k}$$

$$\hat{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|} = \frac{\frac{1}{\sqrt{2}}\left(-\cos t\,\hat{i} - \sin t\,\hat{k}\right)}{\frac{1}{\sqrt{2}}\sqrt{(-\cos t)^2 + (-\sin t)^2}}$$

$$= -\cos t\,\hat{i} - \sin t\,\hat{k}$$



$$\hat{N} = \frac{d\vec{T}/dt}{\left|d\vec{T}/dt\right|}$$

. Notice — if I were increasing speed around circle, $\vec{a}$ would not be perpendicular to $\vec{v}$, but $\hat{N}$ is always perpendicular to $\hat{T}$

proof next page

Fact: Any unit vector $\vec{u}(t)$ is always perpendicular to its derivative ← special case: $\hat{T} \perp \hat{N}$

Pf:

Leibniz / product

$$\vec{u} \cdot \frac{d\vec{u}}{dt} = \frac{d}{dt}(\vec{u} \cdot \vec{u}) - \frac{d\vec{u}}{dt} \cdot \vec{u}$$

$$\Rightarrow 2 \frac{d\vec{u}}{dt} \cdot \vec{u} = \frac{d}{dt}(\vec{u} \cdot \vec{u}) = \frac{d}{dt}\left(|\vec{u}|^2\right)$$

$|\vec{u}| = 1$ unit vector

$$= \frac{d}{dt}(1)$$

$$= 0$$

$\Rightarrow \frac{d\vec{u}}{dt} \cdot \vec{u} = 0$, so orthogonal. ∎

Finally: One other object of geometric significance

Curvature:

$$K = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|$$

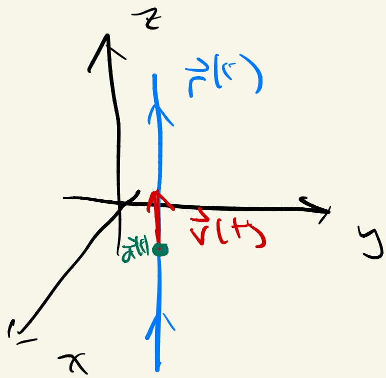
"$\frac{\text{amount of rotation}}{\text{speed}}$"

Ex: Line

$\vec{r}(t) = \hat{i} + \hat{j} + t\hat{k}$

$\vec{v}(t) = \hat{k}$

$\vec{a}(t) = \vec{0}$



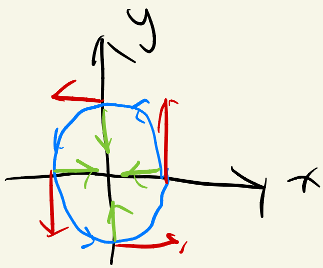
$$\Rightarrow \vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \hat{k}$$

$$\Rightarrow \frac{d\vec{T}}{dt} = 0 \Rightarrow K = 0$$

Ex: Circle

$\vec{r}(t) = \cos(t)\,\hat{i} + \sin t\,\hat{j}$

$\vec{v}(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$



$$\Rightarrow \vec{T} = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{-\sin t\,\hat{i} + \cos t\,\hat{j}}{1} = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

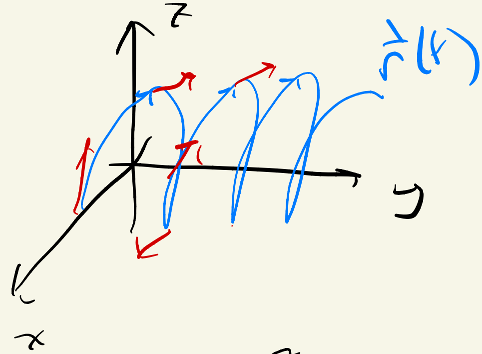
$$\Rightarrow K = \frac{1}{1}\cdot\left|\frac{d\vec{T}}{dt}\right| = |-\cos t\,\hat{i} - \sin t\,\hat{j}| = 1$$

Ex: Helix

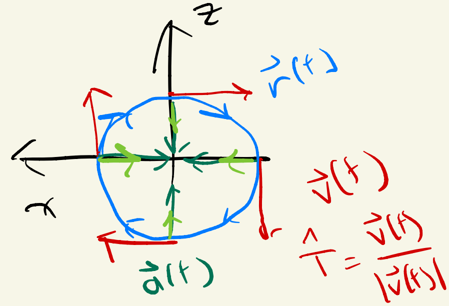
$$\vec r(t) = \cos t\,\hat i + t\,\hat j + \sin t\,\hat k$$

$$\vec v(t) = -\sin t\,\hat i + \hat j + \cos t\,\hat k$$

$$\vec a(t) = -\cos t\,\hat i - \sin t\,\hat k$$



$$\hat T = \frac{-\sin t}{\sqrt 2}\hat i + \frac{1}{\sqrt 2}\hat j + \frac{\cos t}{\sqrt 2}\hat k$$



$$\hat N = \frac{d\vec T/dt}{|d\vec T/dt|} = \frac{\frac{1}{\sqrt 2}(-\cos t\,\hat i - \sin t\,\hat k)}{\frac{1}{\sqrt 2}\sqrt{(-\cos t)^2 + (-\sin t)^2}}$$

$$= -\cos t\,\hat i - \sin t\,\hat k$$

$$\hat N = \frac{d\vec T/dt}{|d\vec T/dt|}$$

$$\Rightarrow \kappa = \frac{1}{|\vec v|}\cdot\left|\frac{d\vec T}{dt}\right| = \frac{1}{\sqrt 2}\left|\frac{-1}{\sqrt 2}\cos t\,\hat i + 0 - \frac{\sin t}{\sqrt 2}\hat k\right|$$

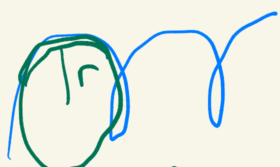
$$= \frac{1}{\sqrt 2}\cdot\frac{1}{\sqrt 2}\cdot 1$$

$= \frac{1}{2}$ ← less than 1, but greater than 0

Interpretation: Radius of curvature

circle

$r = \frac{1}{\kappa} = \frac{1}{1} = 1$



$r = \frac{1}{\kappa} = 2$ (bigger)

$r = \frac{1}{\kappa} = \infty$

infinitely large circle